Monogamy inequality for distributed Gaussian entanglement

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(Dated: November 4, 2006)

We show that for all n-mode Gaussian states of continuous variable systems, the entanglement shared among n parties exhibits the fundamental monogamy property. The monogamy inequality is proven by introducing the Gaussian tangle, an entanglement monotone under Gaussian local operations and classical communication, which is defined in terms of the squared negativity in complete analogy with the case of n-qubit systems. Our results elucidate the structure of quantum correlations in many-body harmonic lattice systems.

PACS numbers: 03.67.Mn, 03.65.Ud

Quantum entanglement, at the heart of quantum correlations and a direct consequence of the superposition principle, cannot be freely shared among many parties, unlike classical correlations. This is the so-called *monogamy* property [1] and is one of the fundamental traits of entanglement and of quantum mechanics itself [2]. Seminal observations on the monogamy property and its precise quantitative statement in mathematical terms are due to Coffman, Kundu, and Wootters [3]. They proved the following inequality for a generic state ρ_{ABC} of three qubits,

$$\tau(\rho_{A:BC}) \;\geq\; \tau(\rho_{A:B}) \;+\; \tau(\rho_{A:C})\,, \tag{1}$$

where τ is an entanglement monotone [4] known as the *tangle* [3], $\tau(\rho_{A:BC})$ stands for the bipartite entanglement across the bipartition $A : BC$, and $\rho_{A:B(C)} = \mathrm{Tr}_{C(B)}\{\rho_{ABC}\}$. Inequality (1) clearly elucidates the restriction on the sharing of entanglement among the three parties. The monogamy inequality (1) holds for the tangle defined through the square of the concurrence [5, 6]. Recently, Osborne and Verstraete [7] have generalized the monogamy inequality to n-qubit systems, proving a longstanding conjecture formulated in Ref. [3], with important consequences for the description of the entanglement structure in many-body spin systems.

For higher-dimensional systems much less is known on the qualification, let alone the quantification of bipartite and multipartite entanglement, the situation worsening with increasing dimension of the Hilbert space due to the exponential increase in the complexity of states. Remarkably, in the limit of continuous variable (CV) systems with an infinite dimensional Hilbert space, if one focuses on the theoretically and practically relevant class of Gaussian states an almost as comprehensive characterization of entanglement has been achieved as in the case of qubit systems [8]. In this context, the natural question arises whether the monogamy inequality holds as well for entanglement sharing in CV systems, and in particular in generic Gaussian states of harmonic lattices. The first step towards answering this question has been taken by Adesso and Illuminati [9] (see also [10]). They proved the monogamy inequality for arbitrary three-mode Gaussian states and for symmetric n-mode Gaussian states, defining the CV tangle or "contangle" as the square of the logarithmic negativity (an entanglement monotone [11]). However, it is known [12] that in two-qubit systems the concurrence is equivalent to another related entanglement measure, the negativity [13], so it appears natural to promote the tangle to CV systems by defining it in terms of the squared negativity itself [14].

In this paper we provide the complete answer to the question posed above. We prove that the monogamy inequality *does* hold for all Gaussian states of multimode CV systems with an arbitrary number n of modes and parties $A_1, \ldots, A_n$, thus generalizing the results of [9, 10]. As a measure of bipartite entanglement, we define the Gaussian tangle via the square of negativity, in direct analogy with the case of n-qubit systems [7]. Our proof is based on the symplectic analysis of covariance matrices and on the properties of Gaussian measures of entanglement [15]. The monogamy constraint has important implications on the structural characterization of entanglement sharing in CV systems [9, 10], in the context of entanglement frustration in harmonic lattices [16], and for practical applications such as secure key distribution and communication networks with continuous variables.

For a $A_1 : A_2 \ldots A_n$ bipartition associated to a pure Gaussian state $\rho^{(p)}_{A:B}$ with $A = A_1$ (a subsystem of a single mode) and $B = A_2 \ldots A_n$, we define the following quantity

$$\tau_G(\rho^{(p)}_{A:B}) = \mathcal{N}^2(\rho^{(p)}_{A:B}). \tag{2}$$

Here, $\mathcal{N}(\rho) = \left(\left\|\rho^{T_A}\right\|_1 - 1\right)/2$ is the negativity [11, 13], $\|\cdot\|_1$ denotes the trace norm, and ρ^{T_A} stands for the partial transposition of ρ with respect to the subsystem A. The functional τ_G, like the negativity $\mathcal{N}$, vanishes on separable states and does not increase under local operations and classical communication (LOCC), i.e., it is a proper measure of pure-state bipartite entanglement [4]. It can be naturally extended to mixed Gaussian states $\rho_{A:B}$ via the convex roof construction

$$\tau_G(\rho_{A:B}) = \inf_{\{p_i,\rho_i^{(p)}\}} \sum_i p_i \tau_G(\rho_i^{(p)}), \tag{3}$$

where the infimum is taken over all convex decompositions of $\rho_{A:B}$ in terms of pure *Gaussian* states $\rho_i^{(p)}$: $\rho_{A:B} = \sum_i p_i \rho_i^{(p)}$. By virtue of the convex roof construction, τ_G [Eq. (3)] is an entanglement monotone under Gaussian LOCC (GLOCC) [15, 17].

Henceforth, given an arbitrary n-mode Gaussian state $\rho_{A_1:A_2\ldots A_n}$, we refer to τ_G [Eq. (3)] as the *Gaussian tangle* and we now prove the general monogamy inequality

$$\tau_G(\rho_{A_1:A_2\ldots A_n}) \geq \sum_{l=2}^{n} \tau_G(\rho_{A_1:A_l}) \,. \tag{4}$$

To this end, we can assume without loss of generality that the reduced two-mode states $\rho_{A_1:A_l} = \mathrm{Tr}_{A_2\ldots A_{l-1}A_{l+1}\ldots A_n}\rho_{A_1:A_2\ldots A_n}$ of subsystems $(A_1 A_l)$ $(l = 2,\ldots,n)$ are all entangled. In fact, if for instance $\rho_{A_1:A_2}$ is separable, then $\tau_G(\rho_{A_1:A_3\ldots A_n}) \leq \tau_G(\rho_{A_1:A_2\ldots A_n})$ because the partial trace over the subsystem A_2 is a local Gaussian operation that does not increase the Gaussian entanglement. Furthermore, by the convex roof construction of the Gaussian tangle, it is sufficient to prove the monogamy inequality for any *pure* Gaussian state $\rho^{(p)}_{A_1:A_2\ldots A_n}$ (see also Refs. [3, 7, 9]). Therefore, in the following we can always assume that $\rho_{A_1:A_2\ldots A_n}$ is a pure Gaussian state for which the reduced states $\rho_{A_1:A_l}$ $(l = 2,\ldots,n)$ are all entangled.

Some technical preliminaries are in order. A n-mode (pure or mixed) Gaussian state ρ is completely characterized by the covariance matrix (CM) $\gamma_{jk} = 2\mathrm{Tr}[\rho(R_j - d_j)(R_k - d_k)] - i(J_n)_{jk}$, and by the displacement vector $d_k = \mathrm{Tr}(\rho R_k)$. Here $R = (\omega_1^{1/2}Q_1, \omega_1^{-1/2}P_1, \ldots, \omega_n^{1/2}Q_n, \omega_n^{-1/2}P_n)^T$, with Q_k and P_k the canonical quadrature-phase operators (position and momentum) for mode k with energy ω_k, and $J_n = \oplus_{j=1}^{n} J_1$ with $J_1 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, the symplectic matrix [8]. Every unitary transformation for general (pure or mixed) Gaussian states $\rho \mapsto U\rho U^\dagger$ is associated to a symplectic transformation $\gamma \mapsto S\gamma S^T$ with $S \in \mathrm{Sp}(2n,\mathbb{R}) = \{S | S J_n S^T = J_n\}$. Positivity of the density matrix ρ is expressed in terms of γ as

$$\gamma + iJ_n \geq 0 \,. \tag{5}$$

Under partial transposition, $\rho_{A_1:A_2\ldots A_n} \mapsto \rho^{T_{A_1}}_{A_1:A_2\ldots A_n}$, the CM γ is transformed to $\widetilde{\gamma} = F_n \gamma F_n$, where $F_n = \mathrm{diag}(1,-1,1,1,\ldots,1)$.

We start by computing the left-hand side of Eq. (4). Since $\rho_{A_1:A_2\ldots A_n}$ is a $1\times(n-1)$ pure Gaussian state, it can be transformed as [18, 19]

$$U\rho_{A_1:A_2\ldots A_n}U^T = \rho_{A_1:A'_2} \otimes \rho_{A'_3} \otimes \ldots \otimes \rho_{A'_n} \tag{6}$$

by a local unitary transformation $U = U_{A_1} \otimes U_{A_2\ldots A_n}$ without changing the amount of entanglement across the bipartition $A_1 : A_2 \ldots A_n$. In the right-hand side of Eq. (6), $\rho_{A_1:A'_2}$ is a pure two-mode Gaussian state (a two-mode squeezed state) and $\rho_{A'_l}$ $(l = 3,\ldots,n)$ are vacuum states. Thus, $\tau_G(\rho_{A_1:A_2\ldots A_n})$ is equal to $\tau_G(\rho_{A_1:A'_2}) = \mathcal{N}^2(\rho_{A_1:A'_2})$. In turn, $\mathcal{N}^2(\rho_{A_1:A'_2}) = (\tilde{\nu}_-^{-1} - 1)^2/4$ [11, 20], where $\tilde{\nu}_-$ denotes the smallest symplectic eigenvalue of $\widetilde{\gamma}_{A_1:A'_2} = F_2\gamma_{A_1:A'_2}F_2$, with $\gamma_{A_1:A'_2}$ being the CM of $\rho_{A_1:A'_2}$. It is easy to compute $\tilde{\nu}_-$; $\tilde{\nu}_- = \sqrt{\det\alpha} - \sqrt{\det\alpha - 1}$, where $\alpha = \left(\begin{smallmatrix} \gamma_{1,1} & \gamma_{1,2} \\ \gamma_{2,1} & \gamma_{2,2} \end{smallmatrix}\right)$ is the CM of the single-mode reduced Gaussian state $\rho_{A_1} = \mathrm{Tr}_{A_2\ldots A_n}\rho_{A_1:A_2\ldots A_n}$. The CM γ of a n-mode *pure* Gaussian state is characterized by the condition [17] $-J_n\gamma J_n\gamma = \mathbb{1}_{2n}$, which implies

$$\det\alpha + \sum_{l=2}^{n} \det\delta_l = 1 \,, \tag{7}$$

where δ_l is the matrix encoding intermodal correlations between mode 1 and mode l in the reduced state $\rho_{A_1:A_l}$ $(l = 2,\ldots,n)$, described by a CM

$$\gamma_{A_1:A_l} = \begin{pmatrix} \gamma_{1,1} & \gamma_{1,2} & \gamma_{1,2l-1} & \gamma_{1,2l} \\ \gamma_{2,1} & \gamma_{2,2} & \gamma_{2,2l-1} & \gamma_{2,2l} \\ \gamma_{2l-1,1} & \gamma_{2l-1,2} & \gamma_{2l-1,2l-1} & \gamma_{2l-1,2l} \\ \gamma_{2l,1} & \gamma_{2l,2} & \gamma_{2l,2l-1} & \gamma_{2l,2l} \end{pmatrix} = \begin{pmatrix} \alpha & \delta_l \\ \delta_l^T & \beta_l \end{pmatrix}. \tag{8}$$

As $\rho_{A_1:A_l}$ is entangled, $\det\delta_l$ is negative [21]. It is useful to introduce the auxiliary quantities $\Delta_l = -4\det\delta_l > 0$. The Gaussian tangle for $\rho_{A_1:A_2\ldots A_n}$ is then written as

$$\tau_G(\rho_{A_1:A_2\ldots A_n}) = \frac{1}{4}(\tilde{\nu}_-^{-1} - 1)^2 = f\left(\sum_{l=2}^{n}\Delta_l\right), \tag{9}$$

$$\text{where } f(t) = (g^{-1}(t) - 1/2)^2, \text{ with } g(t) = \sqrt{t+4} - \sqrt{t}\,. \tag{10}$$

We observe that $f(t)/t$ is an increasing function for $t > 0$ and $f(0) = 0$ so f is a star-shaped function: $f(ct) \leq c f(t)$ for $c \in [0,1]$ and $t \geq 0$ [22]. Therefore, we have $f(t) \leq \frac{t}{t+s} f(t+s)$ and $f(s) \leq \frac{s}{t+s} f(t+s)$ for $t, s \geq 0$ to obtain $f(t) + f(s) \leq f(t+s)$. That is, f is superadditive [23]. Hence [24],

$$f\left(\sum_{l=2}^{n}\Delta_l\right) \geq \sum_{l=2}^{n} f(\Delta_l). \tag{11}$$

Each term in the right-hand side is well defined since $\Delta_l > 0$.

We are now left to compute the right-hand side of Eq. (4), i.e. the bipartite entanglement in the reduced (mixed) two-mode states $\rho_{A_1:A_l}$ $(l = 2,\ldots,n)$. We will show that the corresponding Gaussian tangle is bounded from above by $f(\Delta_l)$, which will therefore prove the monogamy inequality via Eq. (11). To this aim, we recall that any bipartite and multipartite entanglement in a Gaussian state is fully specified in terms of its CM, as the displacement vector of first moments can be always set to zero by local unitary operations, which preserve entanglement by definition. It is thus convenient to express the Gaussian tangle directly in terms of the CMs. Following Refs. [9, 10, 15, 17], the definition [Eq. (3)] for the Gaussian tangle of a mixed Gaussian state with CM $\gamma_{A_1:A_l}$ can be rewritten as

$$\tau_G(\gamma_{A_1:A_l}) = \inf_{\gamma^{(p)}_{A_1:A_l}} \left\{ \tau_G(\gamma^{(p)}_{A_1:A_l}) | \gamma^{(p)}_{A_1:A_l} \leq \gamma_{A_1:A_l} \right\} , \tag{12}$$

where the infimum is taken over all CMs $\gamma^{(p)}_{A_1:A_l}$ of pure Gaussian states such that $\gamma_{A_1:A_l} \geq \gamma^{(p)}_{A_1:A_l}$. The quantities Δ_l and

$\tau_G(\gamma_{A_1:A_l})$ for any l, as well as every single-mode reduced determinant, are $Sp(2,\mathbb{R})^{\oplus n}$-invariants. For each two-mode partition described by Eq. (8), we can exploit such local-unitary freedom to put the CM $\gamma_{A_1:A_l}$ in standard form [25] with $\alpha = \mathrm{diag}\{a, a\}$, $\beta_l = \mathrm{diag}\{b, b\}$, and $\delta_l = \mathrm{diag}\{c_+, c_-\}$, where $c_+ \geq |c_-|$ [21, 26]. The condition [Eq. (5)] for $\gamma_{A_1:A_l}$ is thus equivalent to the following inequalities

$$a \geq 1,\ b \geq 1,\ ab - c_\pm^2 \geq 1\ ; \tag{13}$$

$$\det \gamma_{A_1:A_l} + 1 = (ab - c_+^2)(ab - c_-^2) + 1 \geq a^2 + b^2 + 2c_+c_-. \tag{14}$$

Furthermore, since the state $\rho_{A_1:A_l}$ is entangled, we have [21]

$$(ab - c_+^2)(ab - c_-^2) + 1 < a^2 + b^2 - 2c_+c_-\,. \tag{15}$$

From Eqs. (14) and (15), it follows that $c_- < 0$. In Eq. (12), $\tau_G(\gamma^{(p)}_{A_1:A_l}) = f(4 \det \alpha^{(p)} - 4)$, which is an increasing function of $\det \alpha^{(p)}$, where $\alpha^{(p)}$ is the first 2×2 principal submatrix of $\gamma^{(p)}_{A_1:A_l}$. The infimum of the right-hand side of Eq. (12) is achieved by the pure-state CM $\gamma^{(p)}_{A_1:A_l}$ (with $\gamma^{(p)}_{A_1:A_l} \leq \gamma_{A_1:A_l}$ and $\gamma^{(p)}_{A_1:A_l} + iJ_2 \geq 0$) that minimizes $\det \alpha^{(p)}$. The minimum value of $\det \alpha^{(p)}$ is given by $\min_{0\leq\theta<2\pi} m(\theta)$ [15], where: $m(\theta) = 1 + h_1^2(\theta)/h_2(\theta)$, with $h_1(\theta) = \xi_- + \sqrt{\eta}\cos\theta$, and $h_2(\theta) = 2(ab - c_-^2)(a^2+b^2+2c_+c_-) - (\zeta/\sqrt{\eta})\cos\theta + (a^2-b^2)\sqrt{1-\xi_+^2/\eta^2}\sin\theta$. Here

$$\begin{aligned} \xi_\pm &= c_+(ab - c_-^2) \pm c_-\,, && (16)\\ \eta &= [a - b(ab - c_-^2)][b - a(ab - c_-^2)]\,, && (17)\\ \zeta &= 2abc_-^3 + (a^2 + b^2)c_+c_-^2 \\ &+ [a^2 + b^2 - 2a^2b^2]c_- - ab(a^2 + b^2 - 2)c_+\,. && (18) \end{aligned}$$

Moreover, $m(\pi) \geq \min_{0\leq\theta<2\pi} m(\theta)$ and therefore

$$\tau_G(\gamma_{A_1:A_l}) \leq f(4m(\pi) - 4) = f\left(4\,\zeta_1^2/\zeta_2\right)\,, \tag{19}$$

where $\zeta_1 = h_1(\pi)$ and $\zeta_2 = h_2(\pi)$. Finally, one can prove that (see the Appendix)

$$\Delta_l = -4 \det \delta_l = -4c_+c_- \geq 4\,\zeta_1^2/\zeta_2\,, \tag{20}$$

which, being $f(t)$ [Eq. (10)] an increasing function of t, entails that $f(\Delta_l) \geq f(4\,\zeta_1^2/\zeta_2)$. Combining this with Eq. (19) leads to the crucial $Sp(2,\mathbb{R})^{\oplus n}$-invariant condition

$$\tau_G(\gamma_{A_1:A_l}) \leq f(\Delta_l)\,, \tag{21}$$

which holds in general for all $l = 2 \ldots n$ and does not rely on the specific standard form of the reduced CMs $\gamma_{A_1:A_l}$ [25]. Then, recalling Eqs. (9), (11), and (21), Inequality (4) is established. This completes the proof of the monogamy constraint on CV entanglement sharing for pure n-mode Gaussian states distributed among n parties. As already mentioned, the proof immediately extends to arbitrary mixed Gaussian states by the convexity of the Gaussian tangle [Eq. (3)]. ■

Summarizing, we have defined the Gaussian tangle τ_G, an entanglement monotone under GLOCC, and proved that it is monogamous for all multimode Gaussian states distributed among multiple parties. The implications of our result are manifold. The monogamy constraints on entanglement sharing are essential for the security of CV quantum cryptographic schemes [27], because they limit the information that might be extracted from the secret key by a malicious eavesdropper. Monogamy is useful as well in investigating the range of correlations in Gaussian matrix-product states of harmonic rings [28], and in understanding the entanglement frustration occurring in ground states of many-body harmonic lattice systems [16], which, following our findings, may be now extended to arbitrary states beyond symmetry constraints.

On the other hand, investigating the consequences of the monogamy property on the structure of entanglement sharing in generic Gaussian states along the lines of Refs. [9, 10], reveals that there exist states that maximize both the pairwise entanglement in any reduced two-mode partition, and the residual distributed (multipartite) entanglement obtained as a difference between the left-hand and the right-hand side in Eq. (4). The simultaneous monogamy and *promiscuity* of CV entanglement (unparalleled in qubit systems), which can be unlimited in four-mode Gaussian states [29], allows for novel, robust protocols for the processing and transmission of quantum and classical information [10]. The monogamy inequality [Eq. (4)] bounds the persistency of entanglement when one or more nodes in a CV communication network sharing generic n-mode Gaussian resource states are traced out.

At a fundamental level, the proof of the monogamy property for all Gaussian states paves the way to a proper quantification of genuine multipartite entanglement in CV systems in terms of the residual distributed entanglement. In this respect, the intriguing question arises whether a *stronger* monogamy constraint exists on the distribution of entanglement in many-body systems, which imposes a physical trade-off on the sharing of both bipartite and genuine multipartite quantum correlations. It would be important to understand whether the inequality [Eq. (4)] holds as well for discrete-variable qu*d*its ($2 < d < \infty$), interpolating between qubits and CV systems. If this were the case, the (convex-roof extended) squared negativity, which coincides with the tangle for arbitrary states of qubits and with the Gaussian tangle for Gaussian states of CV systems, would qualify as a universal *bona fide*, dimension-independent quantifier of entanglement sharing in all multipartite quantum systems. In such context, a deeper investigation into the analogy between Gaussian states with finite squeezing and effective finite-dimensional systems, focused on the point of view of entanglement sharing, may be worthy.

Acknowledgements.— We thank Hiroshi Imai and Alessio Serafini for support. GA and FI acknowledge financial support from MIUR, INFN, and CNR-INFM.

Appendix.— In this Appendix we prove the inequality in the right-hand side of Eq. (20). From Eqs. (16) and (17), we have $\xi_\pm^2 - \eta = -(ab-c_-^2)[(ab-c_+^2)(ab-c_-^2)+1-a^2-b^2\mp 2c_+c_-]$. Using Eqs. (13–15), we find $\xi_-^2 > \eta$ so that $\zeta_1 = h_1(\pi) = \xi_- - \sqrt{\eta} > 0$ ($\xi_- > 0$) and $\xi_+^2 \leq \eta$ so that $\xi_+ \leq \sqrt{\eta}$. Here, Eq. (13) implies $\xi_+ \geq c_+(ab - c_-^2 - 1) \geq 0$. However, $\xi_+ = 0$ implies $\det \gamma_{A_1:A_l} = (ab - c_+^2)(ab - c_-^2) = 1$, which means that $\gamma_{A_1:A_l}$ is the CM of a

pure Gaussian state. This can be ruled out from the beginnign and therefore we have $\xi_+ > 0$. Substituting $\sqrt{\eta} \geq \xi_+$ into $\zeta_1 = \xi_- - \sqrt{\eta}$, we obtain

$$0 < \zeta_1 \leq \xi_- - \xi_+ = -2c_- \leq 2\sqrt{-c_+c_-} = \zeta_1'. \tag{22}$$

Next, from Eqs. (13) and (18), we observe $\zeta + (a^2+b^2)(c_+ + c_-)(ab - c_-^2 - 1) = -(a-b)^2[c_+ - c_-(ab - c_-^2)] \leq 0$ to obtain $\zeta \leq -(a^2+b^2)(c_+ + c_-)(ab - c_-^2 - 1) \leq 0$. The last inequality is again due to Eq. (13). Hence, we obtain

$$\begin{aligned} \zeta_2 &= h_2(\pi) = 2(ab - c_-^2)(a^2 + b^2 + 2c_+c_-) + \zeta/\sqrt{\eta} \\ &\geq 2(a^2 + b^2 + 2c_+c_-) + \zeta/\xi_+ = \zeta_2'. \end{aligned} \tag{23}$$

Here, we have used Eq. (13), the inequality $\sqrt{\eta} \geq \xi_+ > 0$, and $a^2 + b^2 + 2c_+c_- \geq 2(ab - c_+^2) > 0$. Now, we observe that: $\xi_+(\zeta_2' - 4) = 2(a^2 + b^2 + 2c_+c_-)\xi_+ + \zeta - 4\xi_+ = -4c_- + 3(a^2 + b^2)c_- - 2a^2b^2c_- + 2abc_-^3 - 2abc_+ + (a^2+b^2)c_+(ab - c_-^2) + 8c_-^2c_+ + 4abc_-c_+^2 - 4c_-^3c_+^2 \geq -4c_-[a^2 + b^2 + 2c_+c_- - (ab - c_+^2)(ab - c_-^2)] + 3(a^2+b^2)c_- - 2a^2b^2c_- + 2abc_-^3 - 2abc_+ + (a^2+b^2)c_+(ab - c_-^2) + 8c_-^2c_+ + 4abc_-c_+^2 - 4c_-^3c_+^2 = -(a^2+b^2)c_- + 2a^2b^2c_- - 2abc_-^3 - 2abc_+ + (a^2+b^2)c_+(ab - c_-^2) \geq -2abc_- + 2a^2b^2c_- - 2abc_-^3 - 2abc_+ + 2abc_+(ab - c_-^2) = 2ab(c_+ + c_-)(ab - c_-^2 - 1) \geq 0$, where we have used Eq. (14). Noting that $\xi_+ > 0$, we obtain $\zeta_2' \geq 4$. Finally, Eqs. (22) and (23), with $\zeta_2' \geq 4$, yield: $\zeta_1^2/\zeta_2 \leq \zeta_1'^2/\zeta_2' = -4c_+c_-/\zeta_2' \leq -c_+c_-$. ■

[1] B. M. Terhal, IBM J. Res. & Dev. **48**, 71 (2004).

[2] D. Bruß, Phys. Rev. A **60**, 4344 (1999); M. Koashi, V. Bužek, and N. Imoto, *ibid.* **62**, 050302(R) (2000); W. Dür, G. Vidal, and J. I. Cirac, *ibid.* **62**, 062314 (2000); A. Wong and N. Christensen, *ibid.* **63** 044301 (2001); K. A. Dennison and W. K. Wootters, *ibid.* **65**, 010301(R) (2001); M. Koashi and A. Winter, *ibid.* **69**, 022309 (2004); C.-S. Yu and H.-S. Song, *ibid.* **71** 042331 (2005); M. Christandl and A. Winter, IEEE Trans. Inf. Theory **51**, 3159 (2005); G. Adesso and F. Illuminati, Int. J. Quant. Inf **4**, 383 (2006).

[3] V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000).

[4] G. Vidal, J. Mod. Opt. **47**, 355 (2000).

[5] S. Hill and W. K. Wootters, Phys. Rev. Lett. **78**, 5022 (1997).

[6] W. K. Wootters, Phys. Rev. Lett. **80** 2245 (1998).

[7] T. J. Osborne and F. Verstraete, Phys. Rev. Lett. **96**, 220503 (2006).

[8] G. Adesso and F. Illuminati, quant-ph/0510052; S. L. Braunstein and P. van Loock, Rev. Mod. Phys. **77**, 513 (2005); J. Eisert and M. B. Plenio, Int. J. Quant. Inf. **1**, 479 (2003).

[9] G. Adesso and F. Illuminati, New J. Phys. **8**, 15 (2006).

[10] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. A **73**, 032345 (2006).

[11] G. Vidal and R. F. Werner, Phys. Rev. A **65**, 032314 (2002); M. B. Plenio, Phys. Rev. Lett. **95**, 090503 (2005).

[12] F. Verstraete *et al.*, J. Phys. A **34**, 10327 (2001); S. Lee *et al.*, Phys. Rev. A **68**, 062304 (2003).

[13] K. Życzkowski *et al.*, Phys. Rev. A **58**, 883 (1998).

[14] Monogamy proofs using the squared logarithmic negativiy hold true for the squared negativity as well, as the latter is a convex function of the former. The converse is false.

[15] G. Adesso, and F. Illuminati, Phys. Rev. A **72**, 032334 (2005).

[16] M. M. Wolf, F. Verstraete and J. I. Cirac, Phys. Rev. Lett. **92**, 087903 (2004).

[17] M. M. Wolf *et al.*, Phys. Rev. A **69**, 052320 (2004).

[18] A. S. Holevo and R. F. Werner, Phys. Rev. A **63**, 032312 (2001).

[19] A. Botero and B. Reznik, Phys. Rev. A **67**, 052311 (2003).

[20] G. Adesso, A. Serafini, and F. Illuminati, Phys. Rev. Lett. **92**, 087901 (2004); Phys. Rev. A **70**, 022318 (2004).

[21] R. Simon, Phys. Rev. Lett. **84**, 2726 (2000).

[22] $f(t)$ is convex for $t \geq 0$, which also implies that f is star-shaped.

[23] A. Marshall and I. Olkin, *Inequalities: Theory of Majorization and Its Applications* (Academic Press, San Diego, 1979).

[24] If we defined the Gaussian tangle for pure Gaussian states as the squared logarithmic negativity (contangle) like in [9, 10], we would have, instead of $f(t)$ in Eq. (10), the quantity $\ln^2[g(t)/2]$ which lacks the star-shape property. It can be confirmed numerically that the function $\ln^2[g(t)/2]$ $(t \geq 0)$ is not superadditive. However this does not imply the failure of the n-mode monogamy inequality for the contangle [9], which might be proven with different techniques than those employed here.

[25] The reduced two-mode CMs $\gamma_{A_1:A_l}$ cannot be all brought simultaneously in standard form [17]. However, our argument runs as follows. We apply $Sp(2,\mathbb{R}) \oplus Sp(2,\mathbb{R})$ operations on subsystems A_1 and A_2 to bring $\gamma_{A_1:A_2}$ in standard form, evaluate an upper bound on the Gaussian tangle in this representation, and derive an inequality between local-unitary invariants, Eq. (21), that is therefore not relying on the specific standard form in which explicit calculations are performed. We then repeat such computation for the remaining matrices $\gamma_{A_1:A_l}$ with $l = 3\ldots n$. At each step, only a single two-mode CM is in standard form while the other ones will be retransformed in a form with (in general) non-diagonal intermodal blocks $\delta_{k\neq l}$. However, the determinant of these blocks (and so Δ_k) and the corresponding two-mode entanglement in the CMs $\gamma_{A_1:A_k}$ are preserved, so the invariant condition Eq. (21) holds simultaneously for all $l = 2\ldots n$.

[26] L.-M. Duan *et al.* Phys. Rev. Lett. **84**, 2722 (2000).

[27] F. Grosshans *et al.*, Nature **421**, 238 (2003); F. Grosshans, Phys. Rev. Lett. 020504 (2005); M. Navascues and A. Acin, *ibid.* **94**, 020505 (2005).

[28] G. Adesso and M. Ericsson, Phys. Rev. A **74**, 030305(R) (2006).

[29] G. Adesso, M. Ericsson, and F. Illuminati, quant-ph/0609178.